

Possible Self-Organised Criticality and Dynamical Clustering of Traffic flow in Open Systems

M. E. Lárrega, J. A. del Río
*Centro de Investigación en Energía,
 Universidad Nacional Autónoma de México,
 A.P.34, 62580 Temixco, Mor. México
 email: antonio@servidor.unam.mx*

Anita Mehta*
*Oxford Physics, Clarendon Laboratory,
 Parks Road, Oxford OX1 3PU, U.K.
 email: a.mehta@physics.ox.ac.uk*

We focus in this work on the study of traffic in *open* systems using a modified version of an existing cellular automaton model. We demonstrate that the open system is rather different from the closed system in its 'choice' of a *unique steady-state density and velocity distribution, independently of the initial conditions*, reminiscent of self-organised criticality. Quantities of interest such as average densities and velocities of cars, exhibit phase transitions between free flow and the jammed state, as a function of the braking probability R in a way that is very different from closed systems. Velocity correlation functions show that the concept of a *dynamical cluster*, introduced earlier in the context of granular flow is also relevant for traffic flow models.

I. INTRODUCTION

The flow of traffic in congested urban conditions is a subject of burgeoning interest in many disciplines at the present time; on the one hand traffic scientists [1], [2] are concerned with the formulation of models which could study and with luck, ease, congestion problems in the real world, while physicists see the subject as an interesting paradigm for complex systems [3]. Typically such studies have considered the behaviour of *closed* systems, that is, systems with periodic boundary conditions which are isolated in the sense that the number of cars is conserved. Here by contrast we focus on the study of open systems where nonequilibrium conditions greatly modify the underlying physics, via the introduction and disappearance of cars at the two ends. Even in the steady state, we find that the construction of the phase diagram is totally different, involving as it does an expansion of the phase space, from the so-called 'fundamental' diagram obtained for the closed version [1].

The present study is based in the context of the extensively studied model of Nagel and Schreckenberg [1], [4]; this involves four cell-updating steps involving braking, stochastic driver reaction, and car movement/acceleration. Most studies including stochasticity, as in the above models, have been for closed systems with periodic boundary conditions, with open systems studied mainly [5] in deterministic models.

Here we study the effect of open boundary conditions (as occurs in actual traffic flow) on a modified Nagel-Schreckenberg model. The modification involves stochastic changes to the car occurring *before* the braking step, to model the behaviour of an 'anticipatory' driver. Our results include i) a qualitative change of the phase diagram, with a *unique* steady state for a given braking parameter R , reached from a variety of initial conditions. This is reminiscent of ideas of self-organised criticality (SOC) [6], introduced earlier in the context of sandpiles. ii) the manifestation of a peak in velocity correlation functions, at specific values of R , reminiscent of the dynamical clustering that has been observed in granular media [7].

The NS model has also been developed recently [8] to show metastable states [9] of very high flow. However we have focused on (a modified versions of) the simpler, classic NS model to show, in a well-studied context, that open boundary conditions induce qualitatively new SOC-like behaviour, as well as interesting aspects of dynamical clustering. These could, in principle, be of conceptually useful relevance to more complex models of traffic flow.

This paper is organised as follows. In the first section we present our model. In the next section we present the results concerning the steady-state regime in the open systems under study. Lastly we discuss our results and compare our predictions with observations on real traffic.

*Present and permanent address: S N Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Calcutta 700 091, INDIA, email: anita@bosen.bose.res.in

II. THE MODEL

In the real world, traffic flow always occurs in open systems, i.e. those where cars are always interchanged between some local environment and its surroundings; thus for example, the number of cars is *not* conserved in general in any section of a highway. However most studies involving cellular automata modelling of such systems have sought to focus on the evolution of traffic in closed systems subjected to periodic boundary conditions. In this study we seek to model more closely some situations in traffic flow by looking at systems with *open* boundaries where, as in reality, the number of cars is not conserved.

We base our model on the Nagel-Schreckenberg [4] cellular automaton model, but with the addition of an important modification involving the order of the operators. Before discussing this, we define the model in its conventional form:

The model consists of a one-dimensional array of cells each of which can be occupied by a car with velocity v between v_{\min} and v_{\max} , with $v_{\min} = 1$ and $v_{\max} \in \{1, \dots, 5\}$. Subject to the non-overlapping of cars, the rules for traffic flow are formulated as follows (we assume that the updating time $t = 1$):

P. Proximity step

For cars $i, i-1$, if $v_i + x_i \leq v_{i-1} + x_{i-1}$, then $v'_i \rightarrow v_i + x_i - x_{i-1} - 1$; else $v'_i \rightarrow v_i$, where the primes represent the updated velocities. In words, this implies that the driver of a car brakes if the car in front is close enough to cause a collision, but not otherwise. Put another way, the driver would like to be at the maximal possible velocity consistent with the avoidance of collisions.

N. Noise step

This reflects the stochastic element which, in the original model, allows for the random deceleration of a fraction R of the cars by one unit of velocity. Thus for example in the case of the i^{th} car, the velocity v_i may either stay the same or, if it is part of the randomly selected fraction R of cars, decrease its velocity by one unit; thus, $v'_i \rightarrow v_i - 1$ (except if $v_i = v_{\min}$).

M. Movement step

This updates in parallel the positions of the cars; once again for the i^{th} car, say, this implies $x'_i \rightarrow v_i + x_i$.

A. Acceleration step

This updates in parallel the velocities of the cars by one unit: thus, $v'_i \rightarrow v_i + 1$ (except if $v_i = v_{\max}$).

We emphasise that the above represents the original form of the model in [1], [4], and now proceed to discuss our modification to it, which involves the order of the operators. Our initial investigations indicated that the order of rules *PNMA* led to several unphysical configurations, whereas the order *NPMA* did not. The reason for this is that with the noise being applied *after* the proximity step, cars are unable to adjust to the noise-reduced velocities of the traffic in front. This could lead to an *artificial* jam, arising from the order of the rules rather than from the real dynamics of the system. Also, importantly, our choice of rules could be said to model the behaviour of *anticipatory* drivers rather than, as in the case of the *PNMA* ordering, *reactive* drivers.

III. THE STEADY STATE IN AN OPEN SYSTEM: RESULTS AND ANALYSIS

In this section we describe both qualitative and quantitative features of our results for the steady state of traffic flow in an open system, as described by the model in the preceding section. First of all, we chose the system size L , randomly generated an initial distribution of car positions and velocities, and then introduced a car with velocity $v = 5$ at the origin at every time step. Next we updated the individual car velocities and positions in accord with the rules of the above model and waited for the system to asymptote to its steady-state density (where we used the χ -squared rule to ensure that this limit was obtained). Finally we recorded the densities and velocities of cars at different positions for use in our later analysis. We mention below some of the specific features of our procedure to ensure convergence to the steady state:

- We chose system sizes L from 200 to 10,000 units, and found that although the time required to reach the steady state was enormous as the system size was increased, the steady-state densities or velocities so obtained did not vary appreciably. In fact we found that for the really large system sizes of say 10,000 units, most of the cars after a distance of ~ 400 units showed the behaviour trivially to be expected of that value of R , i.e. they were either jammed or free-flowing, and thus no longer impacted by the initial car. We thus present in this paper only data obtained for $L = 200, 400$.
- We varied the rules governing the introduction of the initial car, for example, choosing to introduce such a car at alternate rather than consecutive timesteps, and found that this made no significant difference to our results.

- Lastly we varied the 'seed' configurations to do with initial densities and velocities on the line, and found that this made absolutely no difference to our results. The results presented in this work involve averages over 1000 realisations of the experiment.

A. Qualitative results

Our first step is to compare the spacetime diagrams for the case of closed boundary conditions and open ones, on the former of which one of us has carried out extensive investigations [10]. We present below the spacetime diagrams for an open system with $R = 0.7$ in Fig. 1 and a system with periodic boundary conditions with the same R in Fig. 2. We note that while the specification of an initial density by definition determines the final density in the closed system (since cars cannot be 'lost' in the presence of closed boundaries) it does no such thing in the case of the open system, where, in the example shown in Fig. 1, the system evolves from an initially low-density configuration to a jammed state. In some sense we see already the signs that the open system 'chooses' its own final density, while the closed system simply maintains its initially chosen one.

Next we examine the profile of the velocity distribution in the open (Fig. 3a) and closed (Fig. 3b) systems for the *same* initial density and value of R in both cases. For the closed system, we find a relatively larger proportion of high-velocity cars persisting even after a long time has elapsed, compared to the open system, where the number of cars with velocities greater than 1 decays to zero after an initial transient. (It is important to emphasise that the value of the 'most probable' velocity in each case will depend on R). Additionally, while there is a kind of periodicity that is evident in the case of the closed system, with 'waves' of cars of a given velocity appearing and disappearing, separated by local 'spurts' in their value, no such phenomenon is observed in the open system, where the number of cars with velocity 1 gradually increases with time to span the system (although there is an interesting rise in the number of cars with velocity 2, till its decay to zero at $t \approx 500$). We emphasise once again that these examples are chosen only to bring out the differences between the closed and open systems, and that for example a different value of R would result in qualitatively similar but quantitatively different conclusions.

Next, in Figs. 4 and 5, we show that for the open system, initial conditions involving different densities and different randomly generated configurations, all converge to the *unique* densities and velocities characterising the steady state for $R = 0.3$ and 0.7 respectively. We note that the time required by the open system to converge to the steady state is about $10 \times L$, where L is the system size [11], with the exception of the region around the jamming transition, where the transient time can be about $100 \times L$. We show, for comparison, the situation for the closed system in Figure 6; here the initial densities are maintained, and the value of the steady-state velocity depends strongly on the value of the density, *unlike the case of the open system*. Also, in comparison with the open system, the convergence times are virtually instantaneous.

We see thus that in the open system, *arbitrary* initial densities and velocity distributions *evolve towards a unique steady state for a given R* characterised by a final mean density and velocity distribution. The consequences of this apparently simple statement are profound; for example the fundamental flux vs. density diagram obtained in the case of the closed system [12] for a given value of R *collapses to a point* in the open system, since there is only one possible value of density ρ and velocity v in the latter case.

We discuss this unique 'selection' by the open system of steady-state densities and velocities later, but for the present, simply assert that this convergence enables us to work with average densities and velocities (obtained by averaging over time, in the steady state, as well as space, and finally over different initial configurations and noise realisations of the system) in the next subsection.

B. Densities, velocities and correlation functions: a quantitative analysis

We next present and interpret quantitative results on average velocities and densities of cars in the steady state, in addition to examining their fluctuations via correlation functions. In Fig. 7a, the mean density and velocity for systems of size $L = 200, 400$ are plotted as a function of R . As is evident, the curves are coincident, reflecting our contention that the steady state obtained in our work is not system-size dependent beyond about $L = 200$. We see strong evidence of a phase transition which arises around $R_c \sim 0.55$, $\rho_c \sim 0.55$. (These numbers are obtained from an analysis of $\frac{d\rho}{dR}$ vs R , which is shown in Fig. 7b; we will have more to say about the latter graph and its implications later on).

We notice that the density curve is a smooth S-shaped function while that for the velocity is a smooth inverse S-shaped function. Their intersection indicates the likely neighbourhood of the phase transition observed between regions of low ρ and high v ('freely flowing traffic') on the one hand, and regions of high ρ and low v ('congested'

or ‘jammed’ traffic) on the other. Earlier work on closed systems seems to categorise phase transitions in traffic flow as being of first order [12] but we are unable to state this definitively in the context of our finite-size investigations on open systems. In particular the ‘selection’ by the system of steady state densities and velocities for a given value of R is rather reminiscent of the phenomenon of self-organised criticality, [6], where the system organises itself into a unique state for a given value of a parameter. On this basis R would seem to be analogous to a temperature-like variable which then determines the density ρ , whose thermodynamic analogue is the system energy.

However, a deeper examination of this issue is relegated to future work, as for example the shape of $\frac{d\rho}{dR}$ vs R (analogous to the temperature dependence of the specific heat of a thermodynamic system) depicted in Fig. 7b, could equally well represent a second-order transition for a finite system, or for example a kind of lambda transition, reminiscent of the first-order transition in glassy systems [13].

We now turn to the discussion of fluctuations via the analysis of correlation functions. Clearly the $\langle \rho_x \rho_{x'} \rangle$ correlation function is not very informative at least in its ‘bare’ version (i.e. where its value is either 0 or 1 at a site); on the other hand, the $\langle v_x v_{x'} \rangle$ correlation function is meaningful. (Since we look only at the steady state behaviour here, time correlation functions such as $\langle v_t v_{t'} \rangle$ are likewise not meaningful). In Fig. 8 we present the behaviour of this as a function of position, for different values of R . We note that the behaviour is generic, with well-defined first and second neighbour ‘shells’, particularly for values of R well away from the transition point. Additionally, we remark on the specific meaning of such *dynamical* correlations; in analogy with earlier work on granular flow [7], we define a *dynamical cluster* for a given R as being the number of sites which are within the first shell of the velocity correlation function. *The physical import of a dynamical cluster is that it reflects the range over which cars are correlated in their velocities; we observe that the size of a dynamical cluster increases as R decreases.* In other words, as fewer cars face random obstacles, more and more of them develop velocity correlations, i.e. they begin to ‘move together’ in clumps. Returning to the analogy with granular flow, this mirrors the situation found in earlier work [7] where a decrease in external perturbations applied to a granular system causes an *increase* in the size of a typical dynamical cluster of grains.

IV. DISCUSSION

We have examined traffic flow in open systems, and found that the nature of the phase diagram is completely altered with respect to the more usual case of periodic boundary conditions. In particular, the fundamental diagram of flux versus density as a function of the parameter R presented recently for closed systems by Eisenblatter et al [12] collapses to a point in the case of an open system; thus, at a given R , traffic flow in an open system is characterised by a *unique density and velocity distribution, independently of initial conditions.*

This unusual and very robust feature leads us to suggest some thermodynamic analogies for the key quantities in traffic flow in open systems: thus, for example *the thermodynamic analogues of density ρ and braking probability R are respectively energy and temperature.* Following this line of reasoning, we speculate that traffic flow in open systems could either be a paradigm of self-organised criticality, or on the other hand be representative of a first-order phase transition in a finite system. The transition in question, that between jammed and free flow, appears to be characterised by a discontinuity in the analogue of the specific heat as a function of R , i.e. $\frac{d\rho}{dR}$ plotted vs R shows a lambda-transition which could be characteristic either of glassy behaviour or indeed of self-organised criticality.

Various special cases of traffic flow modelled by cellular automata have been examined and found to exhibit self-organised criticality [11]; for example, the case of the outflow region of a big traffic jam under cruise control conditions [14] was found to exhibit this. However, we reiterate that our work is to our knowledge the first to investigate the specific issue of the phase diagram as a function of the braking probability R *under the most general conditions.* Our striking findings regarding *the selection by the system of a unique density and velocity distribution* for arbitrary initial conditions suggest that it may well be a rather general paradigm of self-organised criticality, though future work is in progress to investigate this.

Lastly, we mention that in recent experimental work [9] there has been a suggestion that in addition to the transition between jammed and free flow, there could be a transition to ‘synchronised’ flow where cars neither move freely, nor stay jammed, but continue moving by synchronising their velocities. Our findings with regard to the *dynamical cluster* mentioned in the earlier section appear to be in accord with this, in that dynamical clusters, as discussed earlier in the context of granular flow [7], are clusters whose constituents are strongly correlated in their velocities. We hope to explore some of these issues elsewhere.

V. ACKNOWLEDGEMENTS

AM acknowledges the generous hospitality, over many visits, to the Centro de Investigación en Energía in Temixco, where a large portion of this work was carried out. This work was partially supported by DGAPA-UNAM under project IN117798. We are very grateful to Subodh Shenoy for a careful reading of the manuscript.

-
- [1] K. Nagel and M. Schreckenberg, J. Phys. I (France) **12**, 2221 (1992).
 - [2] *Traffic and Granular Flow*, eds. D.E. Wolf, M. Schreckenberg, and A. Bachem (World Scientific, Singapore, 1996).
 - [3] see for example papers on the asymmetric exclusion process (ASEP), such as C. Godreche, J.M. Luck, M. E. Evans, D. Mukamel, S. Sandow and E. R. Speer, J. Phys. A **28**, 6039 (1995).
 - [4] A. Schadschneider and M. Schreckenberg, J. Phys. A **26**, L679 (1993).
 - [5] B. S. Kerner and W. Konhauser, Phys. Rev. E **50**, 50 (1994).
 - [6] P. Bak, C. Tang and K. Wiesenfeld Phys. Rev. A **38**, 368 (1988).
 - [7] G. C. Barker and Anita Mehta, Phys. Rev. A **45**, 3435 (1992); Anita Mehta and G. C. Barker Phys. Rev. Lett. **67**, 394 (1991); *Granular Matter: An Interdisciplinary Approach*, ed. Anita Mehta (Springer-Verlag, New York, 1994).
 - [8] R. Barlovic, L. Santen, A. Schadschneider and M. Schreckenberg, Eur. Phys. J. B **5**, 793 (1998).
 - [9] B.S. Kerner and H. Rehborn, Phys. Rev. Lett. **79**, 4030 (1997).
 - [10] M. E. Larraga, "Simulacion del Transito Vehicular en Carreteras Usando Automatas Celulares", Thesis Benemerita Universidad de Puebla (Mexico, 1997).
 - [11] T. Nagatani, J. Phys. A **28**, L119 (1995).
 - [12] L. Eisenblatter, L. Santen, A. Schadschneider and M. Schreckenberg, Phys. Rev. E **57**, 1309 (1998).
 - [13] G.S. Grest and S.R. Nagel, J. Phys. Chem. **91**, 4916 (1987); R.M. Ernst, S. R. Nagel and G.S. Grest, Phys. Rev. B **43**, 8070 (1991).
 - [14] K. Nagel and M. Paczuski, Phys. Rev. E **51**, 2909 (1995).

VI. FIGURE CAPTIONS

Figure 1. Spacetime diagram for traffic flow in an open system corresponding to a braking probability $R = 0.7$, and starting with an initial density $\rho_i = 0.2$.

Figure 2. Spacetime diagram for traffic flow in a closed system corresponding to a braking probability $R = 0.7$, and a density $\rho = 0.2$.

Figure 3. Profile of the velocity distribution for traffic flow in a) an open system and b) a closed system corresponding to a braking probability $R = 0.7$, and starting with an initial density $\rho_i = 0.2$.

Figure 4. Plots of the time evolution of the a) density and b) average velocity of traffic in an open system for two initial densities $\rho_i = 0.2$ and $\rho_i = 0.7$, and braking probability $R = 0.7$. Both initial conditions evolve to a single density characteristic of the jammed state.

Figure 5. Same as Figures 4 but with braking probability $R = 0.3$; the final state is, as expected, characteristic of free flow in this case.

Figure 6. Evolution of the time dependent averaged velocity for closed systems with two initial densities $\rho = 0.2$ and $\rho = 0.7$, and braking probability $R = 0.3$. In this case we notice that the final state depends strongly on the (initial) values of the density.

Figure 7. a) The 'fundamental diagram' of traffic flow in open systems; the free-flow to jamming transition occurs in the vicinity of the intersection of the density and velocity curves as a function of braking probability R . b) plot of $\frac{dp}{dR}$ vs R ; note the strong resemblance to the lambda transition in glassy systems. Triangles indicate the results for a system of length $L = 200$ while open circles indicate the data for a system size $L = 400$.

Figure 8. Velocity-velocity correlation functions $\langle v_x v_{x'} \rangle$ corresponding to a range of different values of the braking probability R .